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Cayley–Klein algebras as graded contractions of $so(N+1)$

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Abstract. We study $\mathbb{Z}_2^{\otimes N}$ graded contractions of the real compact simple Lie algebra $so(N+1)$, and we identify within them the Cayley–Klein algebras as a naturally distinguished subset.

1. Introduction

The idea of group contraction was first explicitly introduced by Inönü and Wigner [1], in relation to the study of the non-relativistic limit. In spite of a considerable body of literature (see [2, 3] and references therein), contractions, both of groups and especially of representations, have remained an area full of difficulties. Recently, de Montigny and Patera [4], and Moody and Patera [5] have developed a more general formalism of contractions, making use of the theory of Lie gradings. The crucial property of graded contractions is that they preserve a chosen grading [6] of the Lie algebra L to be contracted. These 'graded contractions' contain as particular cases the Inönü–Wigner contractions, but they go well beyond these cases, and embrace into a unified framework the study of algebra contractions and their finite-dimensional representations.

A nice feature of this approach is that the equations that determine all possible contractions of a Lie algebra L compatible with a given grading can be solved at once for all algebras admitting the same grading. Reference [4] contains tables for the grading groups $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$. An interesting application of this theory [7] is the fact that a family of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded contractions of the algebra $so(5)$ (or of any real form of the complex Lie algebra B_2) contains the kinematical algebras. (See [8], where an implicit use of gradings is also made in relation to kinematical groups.)

In this paper we show how the classical family of (orthogonal) Cayley–Klein groups fits in a very natural way within the family of all $\mathbb{Z}_2^{\otimes N}$ graded contractions of $so(N+1)$. The name Cayley–Klein (CK) is linked with the appearance of these groups within the context of Klein's consideration of most geometries as subgeometries of projective geometry and to Cayley's theory of projective metrics (e.g. see [9, 10]). However, the complete classification of these systems was not given by Klein himself. The $N=2$ case was studied under the name of 'quadratic geometries' by Poincaré, using essentially a modern group-theoretical approach (e.g. see [11]), and the classification for arbitrary dimension N was given in 1910 by Sommerville [12], who showed that there are 3^N different systems in dimension N, each

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corresponding to a choice of the kind of measure of distance between points, lines, ..., hyperplanes being either elliptic, parabolic or hyperbolic.

From a modern point of view, these geometries can be looked at as systems of interlinked sets of symmetric homogeneous spaces associated to a group G. In the general description for arbitrary N [13–15] (see also [16, 17] for $N = 2, 3$), a CK geometry is a system characterized by an $N(N+1)/2$-dimensional real Lie group G and a set of N basic commuting involutions $S^{(k)}$ of the corresponding Lie algebra $\mathfrak{g}$. The generators of $\mathfrak{g}$ invariant under the involutions $(S^{(0)}, S^{(1)}, \ldots, S^{(N-1)})$ span N subgroups of G $(H^{(0)}, H^{(1)}, \ldots, H^{(N-1)})$ which are considered as isotopy groups of a point, line, ..., hyperplane (or $(N-1)$-flat) so that the N symmetric homogeneous spaces $(G/H^{(0)}, G/H^{(1)}, \ldots, G/H^{(N-1)})$ are the spaces of points, lines, ..., hyperplanes. The CK Lie group G itself is determined by N fundamental real parameters $(\kappa_1, \kappa_2, \ldots, \kappa_N)$. Denoting by J_{ij} $(i, j = 0, \ldots, N; i < j)$ a basis of the CK Lie algebra $\mathfrak{g}_{(\kappa_1,\ldots,\kappa_N)}$, we have the commutation relations [13–15]

$$[J_{ij}, J_{lm}] = \delta_{im} J_{lj} - \delta_{jl} J_{im} + \delta_{jm} \kappa_{lm} J_{il} + \delta_{il} \kappa_{ij} J_{jm} \tag{1.1}$$

where $i \leqslant l$, $j \leqslant m$, and the coefficients κ_{ij} are defined by $\kappa_{ij} = \prod_{m=i+1}^{j} \kappa_m$, $i, j = 0, 1, \ldots, N$; $(i < j)$. Each parameter $\kappa_1, \kappa_2, \ldots, \kappa_N$ is related to the kind of measure between two points, lines, ..., hyperplanes, and can be rescaled to either $1, 0$ or -1. For a positive, zero or negative value of each κ_i the corresponding measure is respectively elliptic, parabolic or hyperbolic (hence the number 3^N of essentially different CK systems). Alternatively, these coefficients can be related to the constant curvatures of the canonical connections on the symmetrical homogeneous spaces of points, lines, ..., hyperplanes. A complete description will be given in a forthcoming paper [15], but we only remark here that when $\kappa_1 = 1, 0, -1$ and $\kappa_2 = \kappa_3 = \cdots = \kappa_N = 1$, then the CK space of points $G/H^{(0)}$ is identified to the standard sphere $\mathbb{S}^N$, to the Euclidean space $\mathbb{E}^N$ or to the hyperbolic space $\mathbb{H}^N$ as symmetric homogeneous spaces of constant curvature κ_1 corresponding to the groups (1.1) which are in this case either $SO(N+1)$, $E(N) \equiv ISO(N)$ and $SO(N, 1)$. Therefore, these three well known *Riemannian* symmetric spaces can be thought of as particular cases of the CK scheme, where the distance between points is either elliptic, parabolic or hyperbolic, while all others distances (between lines, ..., hyperplanes) are elliptic.

The aim of this paper is two-fold. First, we define $\mathbb{Z}_2^{\otimes N}$ graded contractions of the real algebras $so(N+1)$ for arbitrary N. Second, we show how the N-dimensional CK algebras appear as a distinguished family of these graded contractions of $so(N+1)$. The order of $\mathbb{Z}_2^{\otimes N}$ is larger than the dimension of $so(N+1)$, so that the number of irrelevant contraction parameters grows rapidly as N increases. The strategy of first solving the contraction equations for a $2^N \times 2^N$ symmetric contraction 'universal' matrix and then disregarding the elements which are irrelevant for the specific case under discussion is perhaps not the best choice. Therefore, we first identify all the relevant parameters for this grading of $so(N+1)$, which fall into three classes, and only then do we write down in an adequate form the relevant contraction equations which are completely solved for a special case. Thus the place occupied by the CK algebras within the family of all graded contractions of $so(N+1)$ can be appreciated very easily.

The paper is organized as follows. The next section presents a brief overview about graded contractions and of the particular $\mathbb{Z}_2^{\otimes N}$ grading we are using for $so(N+1)$. We classify the relevant contraction parameters, and we write the contraction equations. In section 3, we show how the CK algebras appear naturally as a subset of the graded contractions of $so(N+1)$, and we briefly touch upon physical applications.

2. $\mathbb{Z}_2^{\otimes N}$ graded contractions of $so(N+1)$

2.1. Graded contractions

Let us recall briefly the theory of graded contractions of Lie algebras. Suppose L is a real Lie algebra, graded by an Abelian finite group Γ whose product is denoted additively. The grading is a decomposition of the vector space structure of L as

$$L = \bigoplus_{\mu\in\Gamma} L_\mu \tag{2.1}$$

such that for $x \in L_\mu$ and $y \in L_\nu$ then $[x, y]$ belongs to $L_{\mu+\nu}$. This is written as

$$[L_\mu, L_\nu] \subseteq L_{\mu+\nu} \qquad \mu,\ \nu,\ \mu+\nu \in \Gamma\,. \tag{2.2}$$

A *graded contraction* of the Lie algebra L is a Lie algebra L_ε with the same vector space structure as L, but Lie brackets for $x \in L_\mu$, $y \in L_\nu$ modified as

$$[x, y]_\varepsilon := \varepsilon_{\mu,\nu}\,[x, y] \qquad \text{in short hand form} \quad [L_\mu, L_\nu]_\varepsilon := \varepsilon_{\mu,\nu}\,[L_\mu, L_\nu] \tag{2.3}$$

where the *contraction parameters* $\varepsilon_{\mu,\nu}$ are real numbers such that L_ε is indeed a Lie algebra [4]. From antisymmetry and Jacobi identities, one easily gets the *contraction equations*:

$$\varepsilon_{\mu,\nu} = \varepsilon_{\nu,\mu} \tag{2.4a}$$

$$\varepsilon_{\mu,\nu}\,\varepsilon_{\mu+\nu,\sigma} = \varepsilon_{\mu,\nu+\sigma}\,\varepsilon_{\nu,\sigma} \tag{2.4b}$$

for all *relevant* values of indices. Condition (2.4a) means that $\varepsilon_{\mu,\nu}$ can be looked at as a symmetric matrix (the *contraction matrix*, see [4, 5] for more details). Each set of parameters ε which is a solution of (2.4) defines a contraction; two contractions $\varepsilon^{(1)}$, $\varepsilon^{(2)}$ are equivalent if they are related by

$$\varepsilon^{(2)}_{\mu,\nu} = \varepsilon^{(1)}_{\mu,\nu}\,\frac{r_\mu r_\nu}{r_{\mu+\nu}} \tag{2.5}$$

(without summation over repeated indices) where the r's are *non-zero* real numbers which should be thought of as *scaling factors* of the grading subspaces in the Lie algebra.

Even if the contraction parameters associated with *any* pair of elements μ, ν in Γ seem to appear in (2.4), many of them will not, for two reasons.

(1) In the direct sum (2.1) only those L_μ which are proper subspaces must be considered; the set of grading group elements μ actually appearing in the direct sum (2.1) is some subset of Γ. The ε's containing an index μ outside this subset will not appear in the system (2.4).

(2) It could happen that in the non-contracted algebra, all the elements $x \in L_\mu$ commute with the elements $y \in L_\nu$; this situation will be denoted symbolically as $[L_\mu, L_\nu] = 0$. Of course, the parameters $\varepsilon_{\mu,\nu}$ corresponding to $[L_\mu, L_\nu] = 0$ are also completely irrelevant and equations (2.4b) which contain such parameters do not appear.

More details about graded contractions of Lie algebras are given in [4, 18]. The analogous theory for the representations is described in [5].

2.2. A fine grading for $so(N+1)$

In this work we consider the family of graded contractions of $so(N+1)$ which preserve a $\mathbb{Z}_2^{\otimes N}$ fine grading. The algebra $so(N+1)$ has $N(N+1)/2$ generators J_{ab}, with $a<b$; $(a, b=0, 1, \ldots, N)$. The non-zero Lie brackets are

$$a<b<c \qquad \begin{cases} [J_{ab}, J_{ac}] = J_{bc} \\ [J_{ab}, J_{bc}] = -J_{ac} \\ [J_{ac}, J_{bc}] = J_{ab} \end{cases} \tag{2.6}$$

(all Lie brackets involving four *different* indices a, b, a', b' like $[J_{ab}, J_{a'b'}]$ are equal to zero). The standard $(N+1)\times(N+1)$ matrix realization is

$$J_{ab} = -E_{ab} + E_{ba} \qquad a<b \tag{2.7}$$

where E_{ab} is the matrix with a single entry of 1 at row a, column b, and zeros at the remaining entries. Throughout the paper we will use a, b, c, d as indices whenever these are implicitly assumed to appear ordered (as in the generators J_{ab}), and i, j, k, l, m where no ordering is implied.

Let $\mathcal{I}$ be the set of indices $\{0, 1, \ldots, N\}$. We denote by $\mathcal{S}$ any subset of $\mathcal{I}$ and $\chi_{\mathcal{S}}(i)$ the characteristic function over $\mathcal{S}$:

$$\chi_{\mathcal{S}}(i) = \begin{cases} 1 & \text{if } i \in \mathcal{S} \\ 0 & \text{if } i \notin \mathcal{S}. \end{cases} \tag{2.8}$$

We define a linear mapping $S_{\mathcal{S}} : so(N+1) \to so(N+1)$, associated with $\mathcal{S}$ as

$$S_{\mathcal{S}} J_{ab} = (-1)^{\chi_{\mathcal{S}}(a)+\chi_{\mathcal{S}}(b)} J_{ab}. \tag{2.9}$$

The properties of $S_{\mathcal{S}}$ are as follow.

(1) $S_{\mathcal{S}}$ is an involutive automorphism (i.e. it provides a $\mathbb{Z}_2$ grading) of $so(N+1)$.
(2) $S_{\mathcal{S}} = S_{\mathcal{I}\setminus\mathcal{S}}$ (i.e. the automorphism associated to a subset $\mathcal{S} \subseteq \mathcal{I}$ is the same as the one associated to its complement $\mathcal{I}\setminus\mathcal{S}$ in the whole set of indices $\mathcal{I}$).
(3) For any two subsets $\mathcal{S}, \mathcal{S}'$ of $\mathcal{I}$, we have $S_{\mathcal{S}} \cdot S_{\mathcal{S}'} = S_{\mathcal{S}\cup\mathcal{S}'} \cdot S_{\mathcal{S}\cap\mathcal{S}'} = S_{\Delta(\mathcal{S},\mathcal{S}')}$, where $\Delta(\mathcal{S}, \mathcal{S}')$ is the symmetric difference of the subsets $\mathcal{S}, \mathcal{S}'$.

For instance, if $\mathcal{I} = \{0, 1, 2, 3\}$, we have $S_{12} = S_{03} = S_0 S_3 = S_{01} S_{13} = S_{013} S_1 = \cdots$, etc.

From property (1) it follows that the generators J_{ab}, with either both indices or none at all in $\mathcal{S}$, span the $S_{\mathcal{S}}$-invariant subspace of the Lie algebra $L \equiv so(N+1)$, whereas the anti-invariant generators (i.e. those multiplied by -1) are the J_{ab} with exactly *one* index in $\mathcal{S}$. From property (2), the total number of $S_{\mathcal{S}}$ is reduced from the number of subsets of $\mathcal{I}$ to 2^N different involutions. Finally, property (3) shows that all these involutions commute, so that they constitute an Abelian group Γ, isomorphic to $\mathbb{Z}_2^{\otimes N}$. This group is generated by N involutions, for instance

$$S_0, \quad S_{01}, \quad S_{012}, \quad \ldots \quad S_{0\ldots N-1}. \tag{2.10}$$

This particular set of involutions will play a special role, and sometimes we denote them simply as $S^{(k)} \equiv S_{0\ldots k}$, $k = 0, \ldots, N-1$.

It is well known [6] that each set of commuting automorphisms of L determines a grading, so Γ becomes a grading group of $so(N+1)$. A generic element $\mu \in \Gamma$ can be written as a product of powers $\prod_{k=0}^{N-1}(S^{(k)})^{\mu_k}$, where $\mu_k \in \{0,1\}$. Relative to the system of generators (2.10) of Γ, μ can be described by a string $\{\mu_k\} \equiv \{\mu_0\mu_1\dots\mu_{N-1}\}$ of N elements containing 0's and 1's. Not all $\mu \in \Gamma$ are associated with a non-empty subspace L_μ. Using (2.9) one can verify that the basis element J_{ab}, $a < b$ belongs to the grading subspace L_μ:

$$\langle J_{ab}\rangle = L_\mu \qquad \equiv \qquad \mu = \{0\dots0\underset{a}{1}\dots\underset{b}{10}\dots0\} \tag{2.11}$$

where all the 1's are in a contiguous string, starting at the ath position and ending at the $(b-1)$th position, eventually preceded and/or followed by strings of 0's. Therefore, out of the 2^N elements $\mu \in \Gamma$, only those $N(N+1)/2$ elements of the form (2.11) have actual associated grading subspaces L_μ. All subspaces L_μ are one-dimensional and thus this $\mathbb{Z}_2^{\otimes N}$ grading is fine [6].

2.3. *The essential contraction parameters for the fine grading of so(N+1)*

Instead of using the complete string $\{\mu_k\}$ to describe μ when solving (2.4) it will prove very useful to denote the particular μ (2.11) by the pair of indices $\mu \equiv ab$ $(a < b)$ and to speak of a, b as the *indices* of μ. In the following, let $\mu = ab$, $\nu = a'b'$.

The complete string associated to $\mu + \nu$ is as follows.

(1) A string with all zeros only if $\mu = \nu$ (i.e. $a = a'$ and $b = b'$; μ and ν have *both* indices in common).
(2) A string with a single contiguous substring of 1's if μ and ν have a *single* index in common.
(3) A string with two substrings of 1's, separated by 0's, if μ and ν have no indices in common.

Cases (1) and (3) apparently produce grading group elements $\mu + \nu$ without associated grading subspaces. However, the commutation relations (2.6) shows that in cases (1) and (3), and only in these cases, we have identically $[L_\mu, L_\nu] = 0$. So, the only *relevant* contraction coefficients $\varepsilon_{\mu,\nu}$ are those with μ, ν of the form (2.11) and with a *single* index in common. They fall naturally into *three* disjoint subsets,

$$\varepsilon_{ab,ab'} \qquad \varepsilon_{ab,bc} = \varepsilon_{bc,ab} \qquad \varepsilon_{ac,a'c}\,. \tag{2.12}$$

By using the symmetry $\varepsilon_{\mu,\nu} = \varepsilon_{\nu,\mu}$ all these ε can be expressed in terms of the three sets of relevant *essential* contraction parameters:

$$\alpha_{a;bc} \equiv \varepsilon_{ab,ac} \qquad \beta_{ac}^{\,b} \equiv \varepsilon_{ab,bc} \qquad \gamma_{ab;c} \equiv \varepsilon_{ac,bc} \qquad a < b < c\,. \tag{2.13}$$

We sum up by classifying the contraction parameters into disjoint classes.

Class 1. Irrelevant parameters (that do not appear in (2.4)), which consists of all $\varepsilon_{\mu,\nu}$ where at least one of the complete strings of μ, ν or $\mu + \nu$ is not of the form (2.11) (a single contiguous string of 1's).
Class 2. Relevant parameters, which are those $\varepsilon_{\mu,\nu}$ where μ, ν and $\mu + \nu$ have complete strings of the form (2.11) with a single string of 1's. This class could be naturally split into three subclasses.

Class 2a. Elements $\varepsilon_{ab,ab'}$; essential elements $\alpha_{a;bc}$.
Class 2b. Elements $\varepsilon_{ab,bc} = \varepsilon_{bc,ab}$; essential elements β_{ac}^{b}.
Class 2c. Elements $\varepsilon_{ac,a'c}$; essential elements $\gamma_{ab;c}$.

The total number of (relevant and non-relevant) contraction parameters, taking into account only the symmetry (2.4*a*), is $2^N(2^N+1)/2$, but they are reduced beforehand to $3\binom{N+1}{3}$ essential relevant parameters upon which the contraction equations (2.4) have to introduce further relations.

2.4. The contraction equations

Whenever one considers contractions of Lie algebras or of their representations, the major problem is to solve the contraction-defining equations, often consisting of very large systems quite tedious to solve by hand. A computer program was devised to treat these cases [18]. However, in spite of the fact that the grading group $\mathbb{Z}_2^{\otimes N}$ is comparatively large, here we find it much more illuminating to analyse these equations in terms of the splitting of the relevant coefficients into the three classes (2a–2c); the results are simple and transparent enough to justify this approach.

So we set out to solve (2.4*b*):

$$\varepsilon_{\mu,\nu}\,\varepsilon_{\mu+\nu,\sigma} = \varepsilon_{\mu,\nu+\sigma}\,\varepsilon_{\nu,\sigma} \tag{2.14}$$

for all values of the three indices μ, ν, σ giving rise to relevant (class 2) ε's. An elementary analysis shows that for fixed μ, ν, which together determine a unique set of *three* different indices i, k, l (see the comment above (2.12)), the only possibilities for σ leading to relevant ε's in (2.14) are either $\sigma = \mu$ (in which case (2.14) reduces to an identity) or the indices of σ are the index in ν but not in μ, and a new fourth index m, different from i, k, l. So for each set of *four* different indices, i, k, l, m, there is a relevant equation of the kind (2.14):

$$\varepsilon_{\underrightarrow{ik},\underrightarrow{il}}\,\varepsilon_{\underrightarrow{kl},\underrightarrow{lm}} = \varepsilon_{\underrightarrow{ik},\underrightarrow{im}}\,\varepsilon_{\underrightarrow{il},\underrightarrow{lm}} \qquad i, k, l, m \text{ all different} \tag{2.15}$$

where the arrow under each index pair means that these indices should always be put in their natural order (so $\underrightarrow{ik}$ stands for ik if $i < k$, but for ki if $k < i$). Each of the ε's appearing in this equation belong to the relevant classes 2a–2c. Furthermore, (2.15) is invariant under the interchange of the first and the second pairs of indices, $ik \leftrightarrow lm$, so that out of the $4! = 24$ equations associated to each set of four different indices i, k, l, m, only *twelve* equations are different.

The next step through solving these equations is to rewrite them using the natural ordering for the four different indices, say $a < b < c < d$, and then replace all *relevant* ε's by the *relevant essential* elements $\alpha_{a;bc}$, β_{ac}^{b}, $\gamma_{ab;c}$ with $a < b < c$ (see (2.13); e.g. $\varepsilon_{bd,bc} = \varepsilon_{bc,bd} = \alpha_{b;cd}$, $\varepsilon_{cd,bc} = \varepsilon_{bc,cd} = \beta_{bd}^{c}$). The 12 equations are:

$$\beta_{ac}^{b}\,\beta_{ad}^{c} = \beta_{ad}^{b}\,\beta_{bd}^{c} \tag{2.16a}$$

$$\alpha_{a;bc}\,\beta_{bd}^{c} = \alpha_{a;bd}\,\beta_{ad}^{c} \qquad \alpha_{a;bd}\,\alpha_{b;cd} = \alpha_{a;cd}\,\beta_{ac}^{b} \qquad \alpha_{a;bc}\,\alpha_{b;cd} = \alpha_{a;cd}\,\beta_{ad}^{b} \tag{2.16b}$$

$$\alpha_{a;cd}\,\gamma_{bc;d} = \alpha_{a;bc}\,\gamma_{ab;d} \qquad \alpha_{a;bd}\,\gamma_{bc;d} = \alpha_{a;bc}\,\gamma_{ac;d} \tag{2.16c}$$

$$\beta_{ad}^{b}\,\gamma_{ac;d} = \beta_{ac}^{b}\,\gamma_{bc;d} \qquad \beta_{bd}^{c}\,\gamma_{ab;d} = \gamma_{ab;c}\,\gamma_{ac;d} \qquad \beta_{ad}^{c}\,\gamma_{ab;d} = \gamma_{ab;c}\,\gamma_{bc,d} \tag{2.16d}$$

$$\alpha_{a;cd}\,\beta_{bd}^{c} = \alpha_{a;bd}\,\gamma_{ab;c} \qquad \alpha_{b;cd}\,\beta_{ad}^{c} = \beta_{ad}^{b}\,\gamma_{ab;c} \qquad \alpha_{b;cd}\,\gamma_{ac;d} = \beta_{ac}^{b}\,\gamma_{ab;d}\,. \tag{2.16e}$$

There is a single equation involving *only* β's, while all others relate ε parameters belonging to either two subclasses or the three subclasses 2a–2c. This fact, which is emphasized in the presentation of the set (2.16), would be difficult to see by solving the equations for $\varepsilon_{\mu,\nu}$ by means of a computer program, because the special role played by the contraction parameters $\varepsilon_{ab,bc}$ would not be highlighted.

It is therefore clear that the strategy for solving the set of equations (2.16) is first to start with (2.16a), and, for each set of β's solving it, substitute into the remaining equations, which are then considerably simplified. It is not our goal here to provide the general solution of the complete system, but rather to see how the 'simplest' solutions lead exactly to the CK algebras.

3. CK algebras as $\mathbb{Z}_2^{\otimes N}$ contractions of $so(N+1)$

3.1. Quasi-regular solutions to the grading equations

In the general case, the solutions of the contraction equations are classed into 'regular' (all ε's different from zero) and 'non-regular' solutions (some ε's equal to zero). Here we refine this classification and call 'quasi-regular' the solutions with all $\beta_{ac}^{\,b} \neq 0$. The 'simplest' quasi-regular solution of (2.16a) has all $\beta_{ac}^{\,b} = 1$, $a < b < c$. (We shall see in section 3.2 that any contraction with *all* $\beta_{ac}^{\,b} \neq 0$ is equivalent to a contraction with all $\beta_{ac}^{\,b} = 1$, so these are essentially the only quasi-regular solutions.) Fixing $\beta = 1$, equations (2.16) reduce to

$$\alpha_{a;bc} = \alpha_{a;bd} \qquad \alpha_{b;cd} = \gamma_{ab;c} \qquad \gamma_{ac;d} = \gamma_{bc;d} \tag{3.1a}$$

$$\alpha_{a;cd} = \alpha_{a;bd}\,\alpha_{b;cd} = \alpha_{a;bc}\,\alpha_{b;cd} \qquad \gamma_{ab;d} = \gamma_{ab;c}\,\gamma_{ac;d} = \gamma_{ab;c}\,\gamma_{bc;d} \tag{3.1b}$$

$$\alpha_{a;cd} = \alpha_{a;bd}\,\gamma_{ab;c} \qquad \gamma_{ab;d} = \alpha_{b;cd}\,\gamma_{ac;d} \tag{3.1c}$$

$$\alpha_{a;cd}\,\gamma_{bc;d} = \alpha_{a;bc}\,\gamma_{ab;d} \qquad \alpha_{a;bd}\,\gamma_{bc;d} = \alpha_{a;bc}\,\gamma_{ac;d} \tag{3.1d}$$

where $a < b < c < d$.

The first equation in (3.1a) indicates that, as long as $b < c$, $\alpha_{a;bc}$ does not depend on c, we may denote them as a two-index object, $A_{ab} \equiv \alpha_{a;bc}$. Likewise, the third equation in (3.1a) implies that $\gamma_{bc;d}$ is actually independent of b as long as $b < c$, so we may denote it as $C_{cd} \equiv \gamma_{bc;d}$. The remaining equation in the group (3.1a) implies $A_{bc} = C_{bc}$. Replacing $\alpha_{b;cd} = \gamma_{ab;c} = A_{bc}$ in (3.1b)–(3.1d), all these equations reduce to a *single* independent equation for each set of three ordered indices:

$$A_{ac} = A_{ab}A_{bc} \qquad a < b < c\,. \tag{3.2}$$

If we call the *index difference* of the object A_{ab} the *positive* integer $b - a$, then equations (3.2) express those A_{ac} with a given index difference in terms of those with a *smaller* index difference. This means that ultimately only those A_{ab} with index difference equal to 1, $A_a := A_{a-1,a}$, $a = 1, 2, \ldots, N$, will be independent. All other A_{ac} are expressed in terms of A_a as

$$A_{ac} = A_{a+1}A_{a+2}\ldots A_c\,. \tag{3.3}$$

So, we have obtained the following theorem.

Theorem 1. Within the ansatz $\beta_{ac}^{\,b} = 1$, any solution of the contraction equations (2.16) is determined by a set of N independent real numbers, A_a, $a = 1, \ldots N$, and is given by:

$$\begin{aligned} &\text{Class 2a} \qquad \varepsilon_{ab,ac} \equiv \alpha_{a;bc} = A_{a+1} \ldots A_b \\ &\text{Class 2b} \qquad \varepsilon_{ab,bc} \equiv \beta_{ac}^{\,b} = 1 \\ &\text{Class 2c} \qquad \varepsilon_{ac,bc} \equiv \gamma_{ab;c} = A_{b+1} \ldots A_c \,. \end{aligned} \tag{3.4}$$

Those contractions with all $A_a \geqslant 0$ belong to the continuous type, all the others being of the discrete type (the continuous or discrete character of a contraction is defined in [4]). The graded contractions of $so(N+1)$ with all $A_a \neq 0$ and at least one $A_a < 0$ provide the different non-compact real forms $so(p, q)$, with $p + q = N + 1$, of $so(N+1)$.

3.2. Equivalence between quasi-regular solutions

Let us return now to (2.16). Under a general scale change, $J_{ab} \to r_{ab} J_{ab}$, (no sum in repeated indices!) with $r_{ab} \neq 0$, the essential relevant contraction coefficients change as

$$\alpha_{a;bc} \to \alpha_{a;bc} \frac{r_{ab} r_{ac}}{r_{bc}} \qquad \beta_{ac}^{\,b} \to \beta_{ac}^{\,b} \frac{r_{ab} r_{bc}}{r_{ac}} \qquad \gamma_{ab;c} \to \gamma_{ab;c} \frac{r_{ac} r_{bc}}{r_{ab}} \,. \tag{3.5}$$

Now we set out to use the freedom allowed by these changes to reduce each possible solution of the contraction equations to some 'standard' form.

We start with the β's. Let us call the *index difference* of $\beta_{ac}^{\,b}$ the *positive* integer $c - a$. Due to the presence of a third index b (such that $a < b < c$) the index difference of any $\beta_{ac}^{\,b}$ is always greater than or equal to 2, and the number of different $\beta_{ac}^{\,b}$ with the same pair ac is one less than their index difference.

Consider the elements $\beta_{a,a+2}^{\,a+1}$, whose index difference is equal to 2. Each of them can be reduced to 1 (as long as they are different from zero) by adjusting the scale coefficients with the same index difference $r_{a,a+2}$ (i.e. take $r_{a,a+2} = \beta_{a,a+2}^{\,a+1}\, r_{a,a+1}\, r_{a+1,a+2}$, so that, according to (3.5), the new $\beta_{a,a+2}^{\,a+1} \to 1$).

Now consider the elements $\beta_{a,a+3}^{\,a+1}$, whose index difference is equal to 3. By using the same procedure, each of them can be reduced to 1 (as long as they are different from zero) by adjusting the scale coefficients $r_{a,a+3}$ (i.e. take $r_{a,a+3} = \beta_{a,a+3}^{\,a+1}\, r_{a,a+1}\, r_{a+1,a+3}$, so that the new $\beta_{a,a+3}^{\,a+1} \to 1$). By iteration, it is clear that this procedure reduces to 1 all those contraction coefficients $\beta_{a\,c}^{\,a+1}$, as long as they are non–zero. But the remaining β's are not independent, and should still satisfy (2.16*a*). In particular, if all $\beta_{a\,d}^{\,a+1} = 1$, these equations for $a, a+1, a+2$ and d imply

$$\beta_{a,a+2}^{\,a+1}\, \beta_{a,d}^{\,a+2} = \beta_{a,d}^{\,a+1}\, \beta_{a+1,d}^{\,a+2}$$

so that $\beta_{a\,d}^{\,a+2} = 1$. The same procedure with $a, a+2, a+3, d$ now implies $\beta_{a\,d}^{\,a+3} = 1$, and so on. Thus, once the $\beta_{a\,c}^{\,a+1}$ are equal to 1, *all* β's are equal to 1. This justifies the statement made at the beginning of section 3.1.

Now, it remains to exploit the freedom still left to reduce the remaining constants A_a to some standard values. This should be done without spoiling the equalities $\beta_{ac}^{\,b} = 1$. By (3.5), any further scale change with $r_{ac} = r_{ab} r_{bc}$, $a < b < c$, will keep unchanged the values of *all* β's. This means that the scale parameters $r_{a,a+1}$ (with index difference equal to 1) are the only free scaling coefficients, all the others being determined through the relation:

$$r_{ab} = r_{a,a+1} r_{a+1,a+2} \cdots r_{b-1,b} \,. \tag{3.6}$$

The behaviour of the remaining essential relevant contraction coefficients $\alpha_{b;cd} = \gamma_{ab;c}$ under such a scale change is

$$\alpha_{b;cd} \rightarrow \alpha_{b;cd} \frac{r_{bc} r_{bd}}{r_{cd}} = \alpha_{b;cd} \frac{r_{bc} r_{bc} r_{cd}}{r_{cd}} = \alpha_{b;cd} (r_{bc})^2 \qquad a < b < c \tag{3.7}$$

and in particular, the change for $A_a \equiv A_{a-1,a} = \alpha_{a-1;ab}$ is

$$A_a \rightarrow A_a (r_{a-1,a})^2 \qquad a = 1, \ldots, N. \tag{3.8}$$

As there are N free scaling coefficients $r_{a-1,a}$, and N essential contraction parameters A_a for the solution (3.4), it is clear that each A_a can be reduced to the standard values $A_a \in \{1, 0, -1\}$.

We can sum up the results obtained in the previous paragraphs as follows.

Theorem 2. Any quasi-regular solution to the contraction equations (2.16) can be reduced by equivalence to one of the 3^N particular solutions determined in theorem 1 by a family of N values $A_a, a = 1, \ldots, N$, each A_a taking values in the set $\{1, 0, -1\}$.

3.3. The CK *algebras as the quasi-regular contracted algebras of* $so(N+1)$

Once the pertinent solutions to the contraction equations have been determined, the contracted Lie algebra obtained from $so(N+1)$ (see (2.6)) with the contraction parameters ε given in (3.4) has the following non-zero Lie brackets:

$$a < b < c \qquad \begin{cases} [J_{ab}, J_{ac}] = \kappa_{ab} J_{bc} \\ [J_{ab}, J_{bc}] = -J_{ac} \\ [J_{ac}, J_{bc}] = \kappa_{bc} J_{ab} \end{cases} \qquad \kappa_{ab} = \prod_{i=a+1}^{b} \kappa_i \qquad a, b = 0, 1, \ldots, N \tag{3.9}$$

(all Lie brackets involving four *different* indices a, b, a', b' as $[J_{ab}, J_{a'b'}]$ are again equal to zero). It is clear that (3.9) coincides with (1.1), so that the family of quasi-regular graded contractions of $so(N+1)$ leads exactly to the family of CK algebras, and the contraction parameters A_a are to be identified with the geometrical constants κ_a appearing in the CK scheme. The commutation relations (3.9) include the Lie algebras of $SO(p,q)$ $(p+q = N+1)$ when all $\kappa_a \neq 0$, those of the inhomogeneous $ISO(p,q)$ $(p+q = N)$ when, for instance, $\kappa_1 = 0$ but all other $\kappa_a \neq 0$; as well as many other different algebras when more κ_a are equal to zero. It is a satisfying and unifying result to find that all of these groups, *first studied in connection with projective geometry and then with projective metrics, come* out as the more 'regular' family of graded contractions of $so(N+1)$.

It is interesting to inquire about the geometrical meaning of the Inönü–Wigner (IW) type contractions associated to each of the $\mathbb{Z}_2$ subgradings of our grading. Each of these contractions is associated to an involution $S_\mathcal{S}$. The effect of the IW contraction is to perform a graded scale change (by a factor λ) on the generators multiplied by -1 under $S_\mathcal{S}$ (i.e. those with a single index in $\mathcal{S}$) and then to take the limit $\lambda \rightarrow 0$. The effect of this scale change on the parameters $\alpha_{i;jk}$, β^i_{jk} and $\gamma_{jk;i}$ can be described simply as follows.

(1) If either both sets $\{i\}$ and $\{jk\}$ or none at all are contained in $\mathcal{S}$, then the parameters $\alpha_{i;jk}$, β^i_{jk} and $\gamma_{jk;i}$ do not change.

(2) If only one of these sets is contained in $\mathcal{S}$, then the parameters $\alpha_{i;jk}$, β^{i}_{jk} and $\gamma_{jk;i}$ inherit a λ^2 factor (so they vanish in the limit $\lambda \to 0$).

In particular, the IW contractions related to the involutions $S_{\mathcal{S}} \equiv S^{(M)}$ with the particular subsets $\mathcal{S} = \{0, 1, \ldots, M\}$ are the contractions around a point in $G/H^{(0)}$ (for $M = 0$), a line in $G/H^{(1)}$ (for $M = 1$), $\ldots$, a hyperplane in $G/H^{(N-1)}$ (for $M = N - 1$). Due to the ordered nature of the indices in α, β, γ, it is clear that for these particular contractions the β's do not change. The effect on the parameters A_a is to make A_{M+1} equal to zero, while keeping unchanged the remaining contraction parameters, in complete accordance with the geometrical interpretation. The contractions associated to another type of subset of $\mathcal{I}$ (for instance $\mathcal{S} = \{02\}$) would describe the behaviour of a CK geometry near another kind of line, which may be of a different type to the standard one. For instance, a contraction of a Poincaré group around a time-like line in Minkowski space leads to the Galilei group, while a contraction around a space-like line (which is of a different kind) leads to the Carroll group and to a different CK geometry.

But these IW contractions do not exhaust all the possible relations within the CK scheme. For instance, sequences of IW contractions, which cannot be described as the limit of a scale change with factors either 1 or λ, are included in the set of graded contractions.

Let us now illustrate the result of theorem 2 by some examples. We first consider the two-dimensional case. The basis is $\{J_{01}, J_{02}, J_{12}\}$, with commutation relations

$$[J_{01}, J_{02}] = \kappa_1 J_{12} \qquad [J_{01}, J_{12}] = -J_{02} \qquad [J_{02}, J_{12}] = \kappa_2 J_{01} . \tag{3.10}$$

Since $N = 2$, the ensuing CK algebras have a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ grading generated by the automorphisms $S^{(0)}$ and $S^{(1)}$ which act as (from (2.9))

$$\begin{aligned} S^{(0)} &\equiv S_0 : (J_{01}, J_{02}, J_{12}) \longrightarrow (-J_{01}, -J_{02}, J_{12}) \\ S^{(1)} &\equiv S_{01} : (J_{01}, J_{02}, J_{12}) \longrightarrow (J_{01}, -J_{02}, -J_{12}) . \end{aligned} \tag{3.11}$$

These involutions provide the $so(3)$ basis with the following grading:

$$L_{\{01\}} \equiv L_{12} = \langle J_{12} \rangle \qquad L_{\{10\}} \equiv L_{01} = \langle J_{01} \rangle \qquad L_{\{11\}} \equiv L_{02} = \langle J_{02} \rangle \tag{3.12}$$

where the subscripts enclosed in braces (in L_μ) denote the complete string $\{\mu_k\}$, while the subscripts not in braces denote the indices of the relevant μ. The total number of relevant essential contraction parameters is $3\binom{N+1}{3} = 3$ (one α, β, γ each). With $\beta = 1$, the others are related to the constants κ_a as

$$\kappa_{12} = \kappa_2 \leftrightarrow \varepsilon_{\{01\},\{11\}} = \varepsilon_{02,12} \equiv \alpha_{01;2} \qquad \kappa_{01} = \kappa_1 \leftrightarrow \varepsilon_{\{10\},\{11\}} = \varepsilon_{01,02} \equiv \gamma_{0;12} . \tag{3.13}$$

In these and in subsequent expressions, the order of each pair of subindices in ε has always been adjusted either in the lexicographic order when μ is described by its complete string or in the order corresponding to the essential relevant elements when μ is described through its indices.

Now we consider the three-dimensional case, with basis

$$\{J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}\}. \tag{3.14}$$

The $\mathbb{Z}_2^{\otimes 3}$ determines the grading subspaces:

$$\begin{aligned} &L_{\{100\}} \equiv L_{01} = \langle J_{01} \rangle \qquad L_{\{110\}} \equiv L_{02} = \langle J_{02} \rangle \qquad L_{\{111\}} \equiv L_{03} = \langle J_{03} \rangle \\ &L_{\{010\}} \equiv L_{12} = \langle J_{12} \rangle \qquad L_{\{011\}} \equiv L_{13} = \langle J_{13} \rangle \qquad L_{\{001\}} \equiv L_{23} = \langle J_{23} \rangle . \end{aligned} \tag{3.15}$$

In this case, out of the $2^N(2^N+1)/2 = 8(8+1)/2 = 36$ contraction parameters only $3\binom{N+1}{3} = 3\binom{4}{3} = 12$ are the essential relevant ones—the four β's are restricted to 1:

$$
\begin{aligned}
&1 \leftrightarrow \varepsilon_{\{010\},\{100\}} = \varepsilon_{01,12} \equiv \beta^1_{02} \qquad && 1 \leftrightarrow \varepsilon_{\{001\},\{010\}} = \varepsilon_{12,23} \equiv \beta^2_{13} \\
&1 \leftrightarrow \varepsilon_{\{011\},\{100\}} = \varepsilon_{01,13} \equiv \beta^1_{03} \qquad && 1 \leftrightarrow \varepsilon_{\{001\},\{110\}} = \varepsilon_{02,23} \equiv \beta^2_{03}
\end{aligned}
\tag{3.16}
$$

and the four remaining α's and γ's are expressed in terms of the $A_a \equiv \kappa_a$, $a = 1, 2, 3$:

$$
\begin{aligned}
&\kappa_{01} = \kappa_1 \leftrightarrow \{\varepsilon_{\{100\},\{110\}} = \varepsilon_{\{100\},\{111\}}\} = \{\varepsilon_{01,02} = \varepsilon_{01,03}\} \equiv \{\alpha_{0;12} = \alpha_{0;13}\} \\
&\kappa_{02} = \kappa_1\kappa_2 \leftrightarrow \varepsilon_{\{110\},\{111\}} = \varepsilon_{02,03} \equiv \alpha_{0,23} \\
&\kappa_{12} = \kappa_2 \leftrightarrow \{\varepsilon_{\{010\},\{011\}} = \varepsilon_{\{010\},\{110\}}\} = \{\varepsilon_{12,13} = \varepsilon_{02,12}\} \equiv \{\alpha_{1;23} = \gamma_{01;2}\} \\
&\kappa_{13} = \kappa_2\kappa_3 \leftrightarrow \varepsilon_{\{011\},\{111\}} = \varepsilon_{03,13} \equiv \gamma_{01;3} \\
&\kappa_{23} = \kappa_3 \leftrightarrow \{\varepsilon_{\{001\},\{011\}} = \varepsilon_{\{001\},\{111\}}\} = \{\varepsilon_{13,23} = \varepsilon_{03,23}\} \equiv \{\gamma_{12;3} = \gamma_{02;3}\}\,.
\end{aligned}
\tag{3.17}
$$

For the contracted group we get the non-zero commutation relations

$$
\begin{aligned}
&[J_{01}, J_{02}] = \kappa_{01} J_{12} \qquad && [J_{01}, J_{03}] = \kappa_{01} J_{13} \\
&[J_{02}, J_{03}] = \kappa_{02} J_{23} \qquad && [J_{12}, J_{13}] = \kappa_{12} J_{23} \\
&[J_{01}, J_{12}] = -J_{02} \qquad && [J_{01}, J_{13}] = -J_{03} \\
&[J_{02}, J_{23}] = -J_{03} \qquad && [J_{12}, J_{23}] = -J_{13} \\
&[J_{02}, J_{12}] = \kappa_{12} J_{01} \qquad && [J_{03}, J_{13}] = \kappa_{13} J_{01} \\
&[J_{03}, J_{23}] = \kappa_{23} J_{02} \qquad && [J_{13}, J_{23}] = \kappa_{23} J_{12}\,.
\end{aligned}
\tag{3.18}
$$

In summary, we have shown that the Lie algebras of the groups of motions of the N-dimensional CK geometries can be obtained as a naturally distinguished subset of $\mathbb{Z}_2^{\otimes N}$ contractions of the Lie algebra $so(N+1)$, according to the scheme described above. The kinematical groups appear in this formalism as CK groups and they have been studied elsewhere both from the point of view of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded contractions of $so(5)$ [7], and in relation to CK groups and their quantum deformations [16, 17]. The relationship with the present work can be established easily by considering the six possible $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ subgradings of the $\mathbb{Z}_2^{\otimes 4}$ grading we have considered for the case $N = 4$; only two are compatible with the kinematical group requirements. We refer the interested reader to [7] for a more detailed account.

A last question would be the following. Do the contraction parameters correspond to physical parameters? There is indeed a direct connection in some cases. In the usual non-relativistic limit, $1/c^2$ appears as the contraction parameter A_2 associated to time-like lines, while A_1 (associated to points) is to be identified with the curvature of the De Sitter spacetime (either $1/R^2$ or $-1/R^2$) when contracting from the De Sitter to the Minkowski space. Similar interpretations can also be made for other contraction parameters.

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